



Northwest Math Sightings – Sun Sine Math

Students in our math classes legitimately ask us sometimes, “When will I ever need to know this stuff?” It’s a question that has many answers depending on who has asked the question and why. Over the years good teachers develop a skill at fishing out the response that will work for this or that student at this or that moment. Sometimes it concerns an application in “real life,” sometimes it has to do with requirements for the next course down the curricular line, or for tests the student must pass and so forth. My favorite answer, though, is this: “When you understand this math your life will be more interesting. Let me explain...”

Car Pool It’s the beginning of the new school year and my morning to drive the carpool. At 7:20 A.M. I turn from our street onto an arterial headed uphill and facing east. Immediately I’m blinded as the rays of the rising sun blast straight down the road and through my windshield. Ten minutes earlier the sun was below the horizon created by the hill and ten minutes from now it will be comfortably above so that the visor will be useful, but right now the sun glares right down the road, making driving hazardous and left turns across oncoming traffic nearly impossible.

This happens every year at about this time and then again in the spring on our east-west and north-south grid of streets. The rising and setting sun is right in your face in the morning as you travel east and in the evening as you head west. Our perennial overcast means we don’t always notice the alignment but this morning the clouds were gone and I did. It reminded me of shortening days and the coming of fall with winter waiting in the wings. Sunrise this morning, according to the newspaper took place at 6:36 A.M. Tomorrow it will be at 6:37 A.M. By the autumnal equinox, September 23rd this year, the sun will rise at 6:57 A.M. and we will be losing nearly three and a half minutes of sunlight every day. By December 22nd, the winter solstice, we will have to wait until very nearly 8 A.M. to see the sun and it will dip below the horizon at 4:21 P.M.

Sun Sine Whenever we quantify our world, math is there. Quantification makes comparison possible and comparison invites analysis. It is this analysis that leads to understanding (or the satisfying illusion of understanding anyway) and so gives a reason to study math. In the present circumstance, we can keep track of how the photoperiod (the length of time the sun spends above the horizon on any given day) changes as the seasons slip by. When we graph the photoperiod against the days of the year we get a curve that looks pretty familiar. Put a couple of years together and we’re pretty sure there’s something trigonometric going on (See figure 1.).

At our latitude in Washington State, the length of the day changes significantly over the course of the year. We enjoyed a photoperiod of about 16 hours on June 21st, the summer solstice this year, with a sunrise at 4:11 A.M. and sunset at 8:11 P.M. On the equinox we get about 12 hours of light and in the depths of winter, there will be only

eight. That's a gloomy prospect but the possibilities for mathematical analysis give us something cheerful to consider.

As noted earlier, the plot of day length versus date, shown below, reminds us of the sine (or cosine) curve. The correspondence is not perfect; the sine curve crosses the y-axis at zero, reaches a maximum height of one, a minimum of negative one and goes through one complete cycle over 2π radians or 360 degrees. Relative to the sine curve, the day length curve is shifted both horizontally and vertically. It crosses the y-axis just after its minimum and it has no negative values. (What would it mean to have a photoperiod less than zero?!) Also the curve runs from a minimum of eight hours to a maximum of 16 hours so that its amplitude is four hours. We can, of course, modify the coefficients of a general equation of the sine curve in order to model the photoperiod. Technology makes it easy, and what an opportunity to construct understanding of the effect that each coefficient has on the graph!

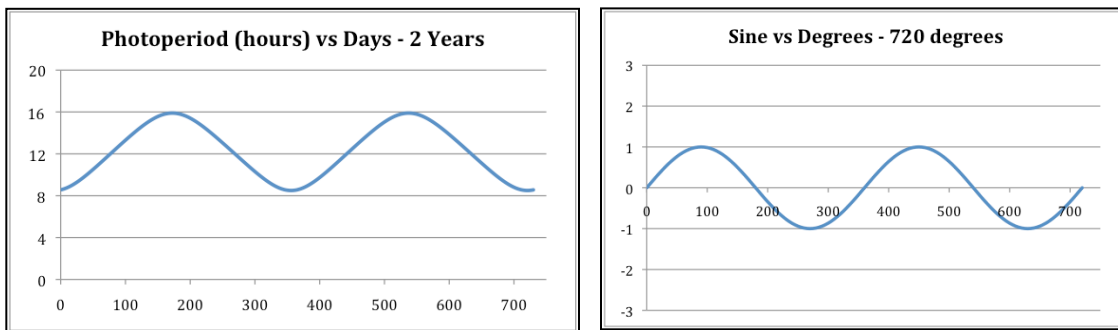


Figure 1a. Photoperiod over two years

1b. Sine curve over 720 degrees

$$y = A \cdot \sin(Bx - C) + D$$

Using the online utilities provided by, for example, the “Trigonometric Graphing” applet on the NCTM Illumination site (<http://illuminations.nctm.org/ActivityDetail.aspx?ID=174>) students can start with a basic sine curve and then select changes to the coefficients and see how the graph responds. Even simpler, using the on-line applet available at <http://www.analyze-math.com/trigonometry/sine.htm> they can drag on-screen sliders to see how changes in “A,” “B,” “C,” and “D” affect the shape and position of the curve. Neither of these have enough range to allow the changes we need to model the photoperiod, but now that I know how the coefficients work I can turn to a graphing calculator or to online versions of the same (e.g., http://my.hrw.com/math06_07/nsmedia/tools/Graph_Calculator/graphCalc.html). Before long I have an equation, $y = 4 \cdot \sin(.97x - 78) + 12$ that provides a reasonable model of the photoperiod with sequential days of the year on the x-axis and y values representing hours of daylight. (See figure 2.) It's not perfect but the amplitude of 4 and the vertical shift of 12 give it a minimum of 8 hours and a maximum of 16 hours which works fairly well for our latitude here in Washington state. Letting C equal 81 shifts the whole curve horizontally so that the graph starts just beyond its minimum (just as the winter solstice occurs about ten days before the start of the new year) and a B value of .99 adjusts the period of the function to be more like 365 days when

the function is plotted using degrees rather than radians. (The values of B and C could be refined but are adequate for this application.)

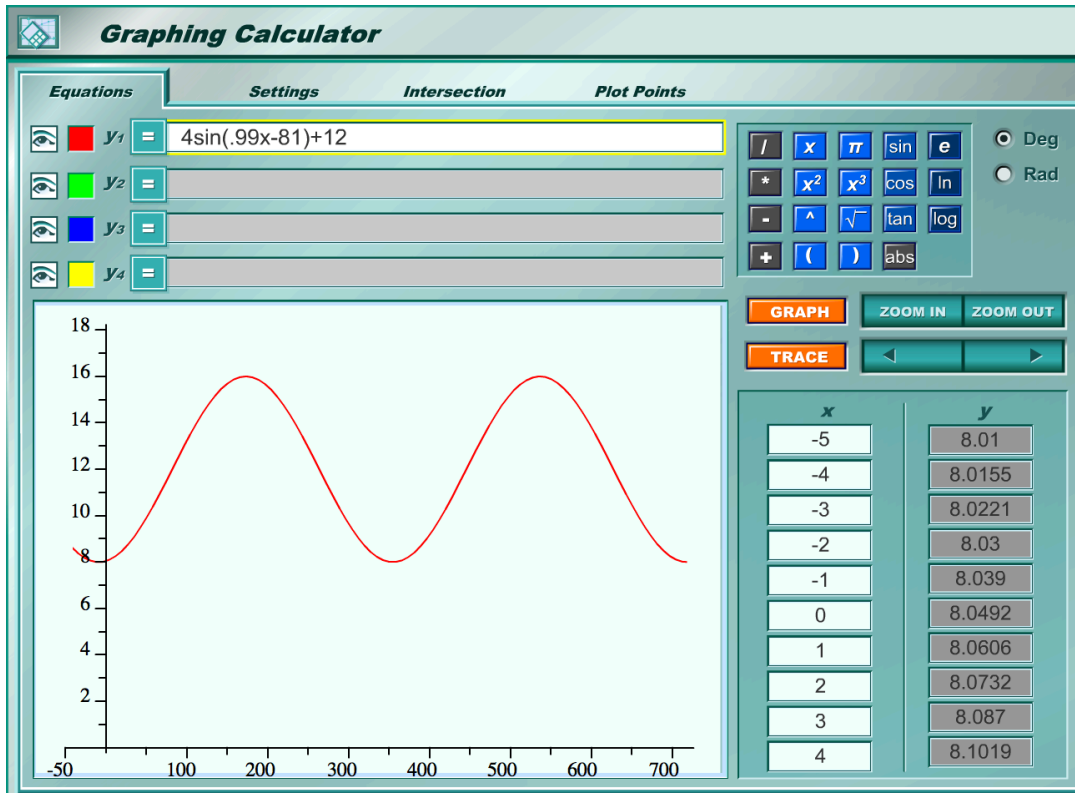


Figure 2. Using a sine function to model the photoperiod over two years

Having this equation enables prediction and further analysis. For example, the slope of the curve tells me how quickly the days' lengths are changing at that point in the year. As I write this we are nearing mid September, 258 days into the year. Looking at the graph for the photoperiod I see that the slope is steep at this time of the year. The change is tied in to the change in the seasons. While there are about 377 hours of daylight in September, October has only 338, and it has one more day than does September! That change is equivalent to about two solid days of darkness.

Elevation Another aspect of our environment, and one that goes a long way toward determining the temperature in the long run, is the elevation of the sun in the sky. At our latitude the sun may be high overhead at noon in the summer time but by the middle of December the sun never gets above an angle of 20 degrees above the horizon. This matters because when the sun is lower in the sky its rays strike us at an angle and are more spread out as a result. With some simple use of right triangles and trigonometric relationships we can see, for example, that a beam of sunshine one mile wide coming in at an elevation angle of 30 degrees is spread out over a horizontal surface that is 2 miles wide. (See figure 3.) When that same beam comes in from straight overhead (90 degrees), it is distributed out over one mile, making it more concentrated and therefore more effective as a source of heat.

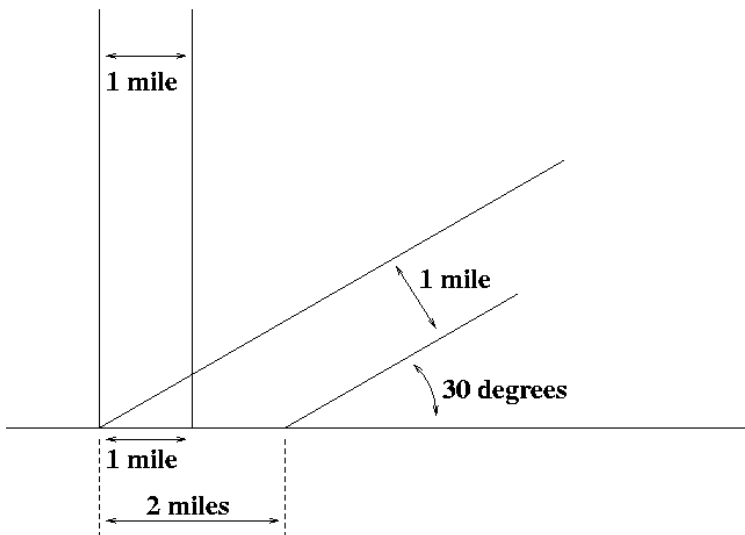


Figure 3. Concentration of the sun's energy varies with the angle of elevation

There are sites online that provide the sun's maximum elevation angle at any day for any location. (See, for example, Star Walk and Sky Live on the iPad or <http://www.srrb.noaa.gov/highlights/sunrise/azel.html> on the Web) Using these I collect some data and make a chart showing the relationship between maximum solar elevation and days of the year. When I do I see that the basic shape of the curve is similar to that of the photoperiod. (See figure 4.) In June the sun rises earlier, stays up later, and ascends to a much greater angle, providing radiation that is less dispersed and therefore more effective at heating the earth, and us! Our hottest days tend to come later in the year just as our coldest nights lag behind the winter solstice but this is because it takes time for the planet and its oceans to heat up and to cool down.

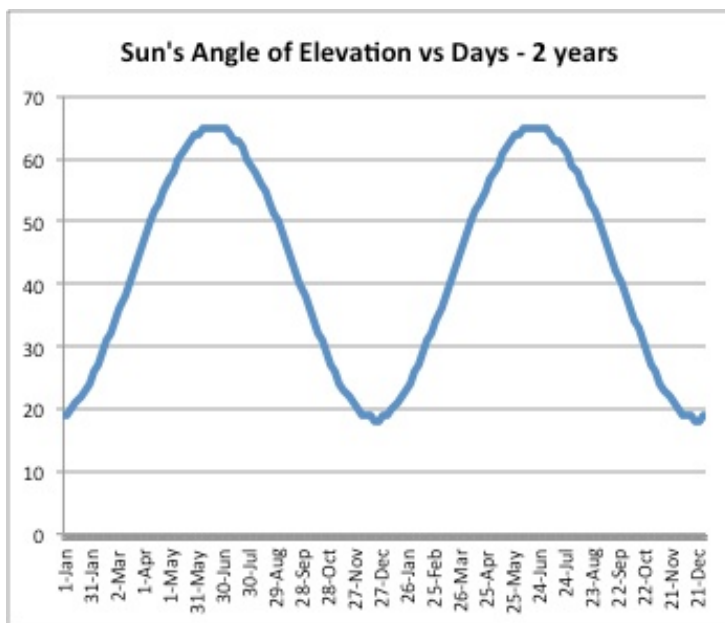


Figure 4. *Solar Elevation - Two years*

I notice, as I enter the data to create the scatter plot in figure 4, that the solar elevation tends to change by about two degrees every five days. This is true during the times of the year when the slope of the curve is steepest, either positive or negative. It is only in the peaks and valleys of the curve that the rate of change slows, turns around, and heads in the other direction. It reminds me of the tide, flooding in, going slack, and then turning back to the sea, following the moon that revolves around the earth as it circles the sun ...

Conclusion I do not need mathematics to appreciate the warmth of the summer sun or the lengthening of days as spring approaches. Yet I pay more attention and I attend differently to these aspects of my environment because I have considered them in a quantitative sense. I see more connections than I might have otherwise and I come to believe that I may be able to create more. In the end, maybe that's why we need to know this stuff.

Mark Roddy
Seattle University
College of Education
mroddy@seattleu.edu
(206) 296-5765