Northwest Math Sightings – Tree Ring Math

Students in our math classes legitimately ask us sometimes, “When will I ever need to know this stuff?” It’s a question that has many answers depending on who has asked the question and why. Over the years good teachers develop a skill at fishing out the response that will work for this or that student at this or that moment. Sometimes it concerns an application in “real life,” sometimes it has to do with requirements for the next course down the curricular line, or for tests the student must pass and so forth. My favorite answer, though, is this: “When you understand this math your life will be more interesting. Let me explain…”

A tree, a large Douglas Fir, had fallen out at the cabin. I saw it lying at the end of the lane as I drove up and was immediately thankful that it had not fallen across the road or onto the cabin itself. I enlisted the aid of a neighbor, Frank, and with his chainsaw we limbed and trimmed until we had a mostly bare trunk lying before us. Twenty inches across at the base, it tapered to a spindly top that had broken off in the fall. I paced along its length, estimating about two and a half feet per stride, and found that the tree had been a little over 100 feet tall! We worked for the better part of the next day to get it cut up for firewood. Along the way I asked Frank to cut me a “tree cookie,” a slice about two inches thick taken near the base of the trunk. I wasn’t sure what I would do with it but something would come up.

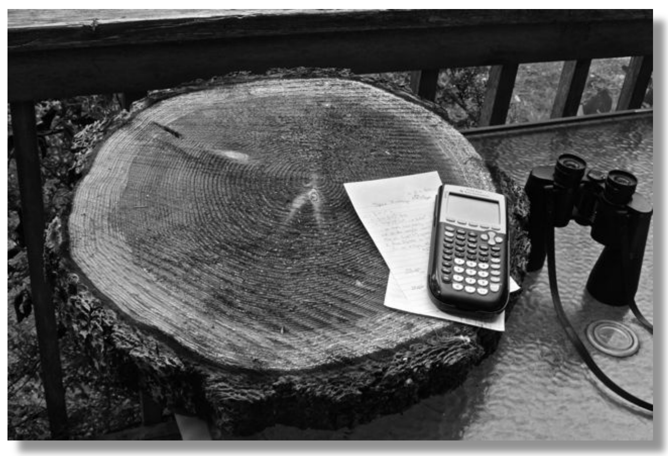


Figure 1. The tree cookie

Weeks later, there it was, sitting on the table on the deck where I had left it. Taking a closer look, I counted about 70 concentric rings, indicating as many years of growth. The rings told an interesting story. That tree must have sprouted sometime around 1941. World War II was about to arrive at Pearl Harbor. FDR was president. Rings close together meant that not much growth had taken place in that year. Fat rings meant good years for growing. But clearly the farther out you went from the center the closer the rings were to one another. Did growth slow down as the tree aged? Thinking further I realized that while the rings were closer together in the later years, the tree at that time was taller so that the overall volume and therefore mass or weight added to the trunk each year might be the same. It might even be greater, … or less.

This needed investigation. I found a tape measure and a pad of paper. I counted out five rings from the center and measured the distance - seven eighths of an inch. That meant that the tree at five years old had been a little short of two inches in diameter. I could see, though, that the rings were not perfect circles and I would get different measurements depending on which direction I headed out from the center. Perhaps the tree grew more vigorously on one side than another. What caused that? Was it the sun shining mostly on the south-facing side? Was it the wind blowing predominantly from one direction or another? Saving those questions for later, I decided to measure out from the center in four different directions, each 90 degrees from the last, and to take the average of the four as the radius of the trunk at that time. I did this for five, 10, 20, 30, 40, 45, and 70 years. The measurements are shown in table 1.

	5 years	10 yrs	20 yrs	30 yrs	40 yrs	45 yrs	70 yrs
Radius 1	$\frac{7}{8}$	$1\frac{7}{8}$	$4\frac{1}{8}$	6	$7\frac{1}{4}$	$7\frac{3}{4}$	$10\frac{1}{4}$
Radius 2	$\frac{7}{8}$	$1\frac{3}{4}$	$3\frac{5}{8}$	$4\frac{11}{16}$	$5\frac{7}{8}$	$6\frac{1}{4}$	$9\frac{1}{4}$
Radius 3	$\frac{15}{16}$	$1\frac{13}{16}$	$3\frac{9}{16}$	5	$6\frac{1}{4}$	$6\frac{3}{4}$	$9\frac{1}{8}$
Radius 4	$1\frac{1}{8}$	$2\frac{5}{16}$	$4\frac{7}{8}$	7	$8\frac{1}{2}$	9	12
Conveniently rounded **average radius**	$\frac{15}{16}$	2	4	$5\frac{11}{16}$	7	$7\frac{1}{2}$	$10\frac{1}{4}$

Table 1. Radius measurements and their approximate averages in inches

So I concluded that at five, 10, 20, 30, 40, 45 and 70 years the trunk had a radius of about $\frac{15}{16}$, 2, 4, $5\frac{11}{16}$, 7, $7\frac{1}{2}$ and $10\frac{1}{4}$ inches. Getting back to my question about how the tree had put on weight over the years, I thought about it this way: each year the tree added a new layer – a new coat over the old - and the tree was always thicker at the base and thinner at the top. I decided that I could approximate the shape of the trunk as a very tall cone. Each year the trunk got a little taller and a little thicker so the cone got a little bigger.

The volume of a cone is given by the expression $\frac{1}{3}\pi r^2 h$ where r is the radius of the cone at the base and h is the cone's height. At five years of age, for example, the radius was a little under an inch … but what was the height? The tree had been five years old in 1946. Over a decade would pass before I would be born. Still there must be a way to estimate the height given the diameter.

Figure 2. Tree cookie with rings for 10, 20, 30, and 40 years emphasized.

I did some research. It turns out that the heights of trees, and Douglas Firs in particular, depend on a good deal more than how long they have been growing. As you would expect, the height they attain will depend on their growing conditions. Also, it seems that they tend to grow more slowly for the first five years, then more rapidly for the next fifty or so years and then return to a slower rate of growth. They can grow for more than 1000 years and attain heights over 300 feet. From ArborDay.org's tree guide [http://www.arborday.org/trees/treeGuide/] I learned that the Douglas Fir is considered a medium growth tree and that such a tree, all other things being equal, can be expected to grow between 13 – 24 inches per year.

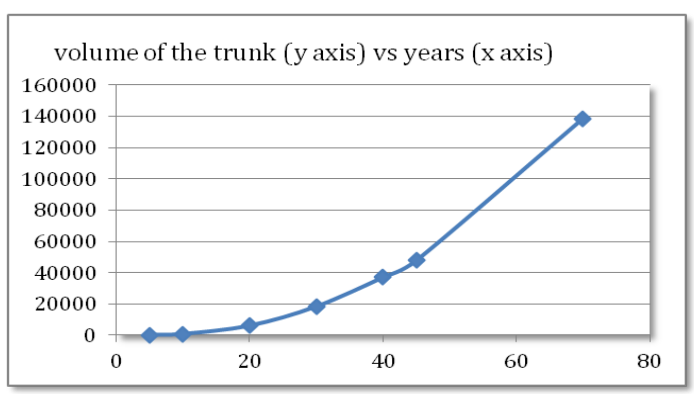
That gave me a way to get a rough estimate of the tree's height for any given year. I took 18 inches per year to be the approximate middle of the growth range given by ArborDay.org. It was a convenient number, a foot and a half per year. So now I could calculate an "h," a height, to go with my "r," the radius I had measured from the tree cookie. That meant that I could get a rough estimate of the volume of the tree - the trunk at least - for any year using the formula for the volume of a cone. Then with a little subtraction I could calculate the change in volume that had taken place at different points in the tree's history.

Table 2 shows the estimates of the tree's height, measured radius, and volume of the tree's trunk as a function of years of growth.

Years	**5 yrs**	**10 yrs**	**20 yrs**	**30 yrs**	**40 yrs**	**45 yrs**	**70 yrs**
Estimated height (feet)	7.5	15	30	45	60	67.5	105
Approx radius (inches)	0.94	2.00	4.00	5.69	7.00	7.50	10.25
Approximate volume of trunk (cubic inches)	83	754	6,032	18,292	36,945	47,713	138,627

Table 2. Estimated height, measured radius, and calculated volume of the tree trunk in cubic inches

The results were a little surprising and I checked them a couple of times. Could the tree trunk go from zero to 83 cubic inches in the first five years and then gain about 670 additional cubic inches in the next five years? If it had a volume of about 750 in^3 after 10 years then, growing at the same rate it would have about 1,500 in^3 after 20 years but my estimate put it at over 6,000 in^3. Wow! Further subtraction showed that the trunk had gained, in successive ten year intervals, 754 in^3 (0-10 years), then 5278 in^3 (10-20 years), then 12,260 in^3 (20-30 years), then 18,653 in^3 (30-40 years), and then 101,632 in^3 in the final 30 years. I went to my computer, started a spreadsheet and made a chart of volume as a function of time. (See fig.2) That



was certainly not a straight line. The trunk was gaining volume at a rate that accelerated over time.

Figure 2. Approximate volume of trunk vs. years

Giving this a little thought, I could see that it made sense. Trees produce glucose, their food, through photosynthesis. Photosynthesis happens in the leaves (needles in this case). The needles are on the braches. The taller the trunk the more branches. The more branches, the more food and so on, so it makes sense that the rate of growth would accelerate, at least for a while. Of course that can't go on forever. The mass of the tree was enormous. Wouldn't it eventually collapse under its own weight? How much *did* it weigh? Knowing the volume of the trunk, I should be able to get a rough estimate if I knew how much each cubic inch weighed. I know that wood floats so it's less dense than water and I know that water weighs one gram per cubic centimeter. Guessing that the fir tree's wood, which is considered to be a "soft" wood, might be half as dense as water (not far off, actually; I looked it up later!) and knowing that there are 2.54 centimeters in every inch, I worked up a little conversion and, with the help of my calculator, found that 138,627 in^3 is equivalent to 2,271,690 cm^3. If each cubic centimeter weighed .5 grams, that made about 1136 kg. Since a kilogram is about 2.2 pounds, that's about 2,500 lbs, well over a ton! The trunk was about 100 feet tall. If we cut it into pieces that were 1.5 feet long, then split each of these into an average of four pieces, that would give about 265 pieces. Dividing 2,500 lbs by 265 pieces told me that the average weight of my chunks of firewood would be about nine and a half pounds. (An average high school math text, by the way comes in at about three or four pounds.)

Looking again at the steeply rising curve in my chart and remembering that the tallest Douglas Firs weren't too much more than 300 feet tall, I wondered what happens in trees to slow down their growth over time. Do their cellular processes simply become less efficient after 100 years or so? (At 52 years of age I feel that I am beginning to develop some personal understanding of this decline.)

Finally I returned to the tree cookie. The rings *were* closer together as the tree got older but the area in each successive ring, being dependent on the *square* of the radius, was increasing more rapidly as the tree grew older. The volume of the trunk, as modeled by a cone, needed only a little change in radius to produce a big change in volume.

The tree rings did tell a story about the tree and the story was made more interesting and more important to me with the aid of some simple mathematics. Looking up from the tree cookie I could see dozens of other trees, still living and growing, and an infinite number of other instances throughout the forest calling out or waiting quietly for my attention, my appreciation, and, sometimes, my analysis.

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